M75 / Burger ch. 3 ①

3.3 → Derivative rules made easy.

From 3.2 Homework, a common Problem was to find the derivative of something like:

$$\frac{d}{dx}\frac{5}{\sqrt{x}}.$$

$$= \lim_{h\to 0} \frac{\frac{5}{\sqrt{x+h}} - \frac{5}{\sqrt{x}}}{h} = 5\cdot\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= 5\cdot\lim_{h\to 0}\frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\cdot\sqrt{x}\sqrt{x+h}\,(\sqrt{x}+\sqrt{x+h})}$$

$$= 5\cdot\lim_{h\to 0}\frac{x-(x+h)}{h\cdot\sqrt{x}\cdot\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$= 5\cdot\lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-5}{x(2\sqrt{x})} = \frac{-5}{2x^{3/2}}$$

Today we will learn the

Power Rule $\frac{d}{dx}x^n = nx^{n-1}$

making the Above much easier.

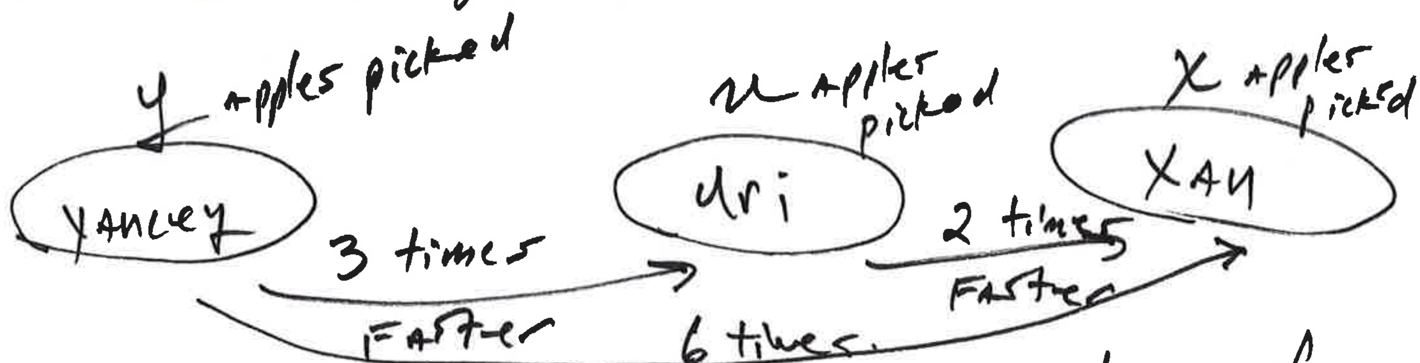
$$\frac{d}{dx}(5x^{-1/2}) = -\frac{1}{2}\cdot 5x^{-\frac{1}{2}-1} = \frac{-5}{2}x^{-3/2} = \frac{-5}{2x^{3/2}} \checkmark$$

3.7 The Chain Rule

Given $\frac{d}{dx}(5x+4)^{100}$, who would want to expand this?

The Chain Rule in words.

Topic: Picking apples.

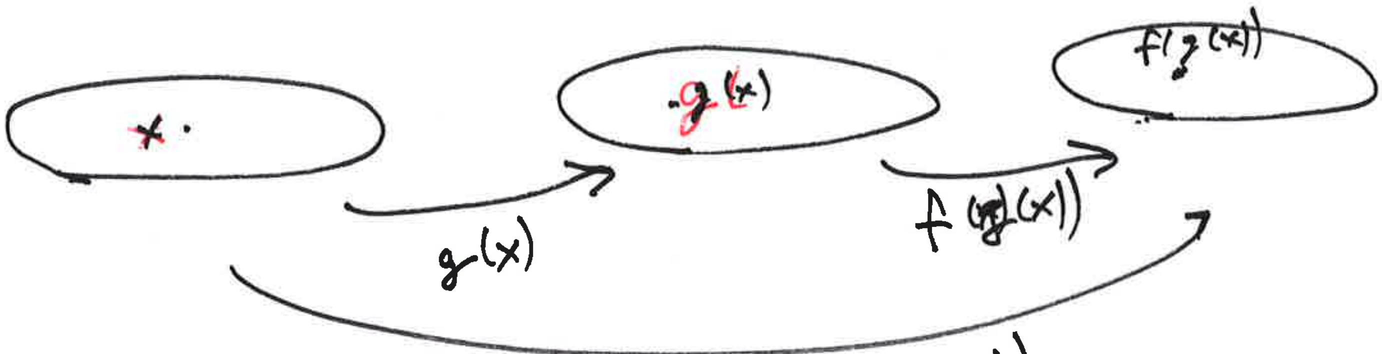


What is $\frac{dy}{dx}$ or the rate of apples picked by Yancey as compared to Xan?

Say Xan picks 5 apples in a minute. Then Uri picks ... 10 and Yancey picks ... 30 in the same minute. ... notice $30 = 6 \cdot 5 = 3 \cdot 2 \cdot 5$.

or $\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$ The Chain Rule Rules!!

Another way to think of the chain rule is:



$$(f \circ g)(x) = f(g(x))$$

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

like $3 \cdot 2 = 6$!

OR; $$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

ex. we have seen $\frac{d}{dx}(x^2) = 2x$ and

$$\frac{d}{dx}(5x+1) = 5$$

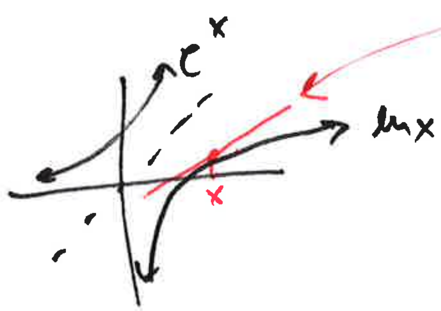
$$\frac{d}{dx}(5x+1)^2 = 2(5x+1)^1 \cdot 5 = 10(5x+1) = \boxed{50x+10}$$

The Derivative of $\ln x$.

- What is $\boxed{\ln x \iff \log_e x\ ?}$

$$\frac{d}{dx}[\ln x] = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

just slope!



$$\boxed{\begin{aligned}\log a - \log b &= \log \frac{a}{b} \\ \log a^r &= r \log a\end{aligned}}$$

$$= \lim_{h\to 0} \frac{\ln \frac{x+h}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \ln\left(1 + \frac{h}{x}\right) = \lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

Now let $u = \frac{h}{x}$ then so $\boxed{\begin{aligned} h &= ux \\ \frac{1}{h} &= \frac{1}{ux}\end{aligned}}$

$h \to 0$ same $u \to 0$

$$= \lim_{u\to 0} \ln(1+u)^{\frac{1}{ux}} = \lim_{h\to 0} \ln\left((1+u)^{\frac{1}{u}}\right)\frac{1}{x}$$

$$= \frac{1}{x} \lim_{u\to 0} \ln(1+u)^{\frac{1}{u}}$$ But By def.

$$\lim_{u\to 0} (1+u)^{\frac{1}{u}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e \approx$$

2.7182818 . . . An irrational #.

can't be expressed as a fraction!

So to conclude . . .

$$\frac{d}{dx}[\ln x] = \frac{1}{x}\cdot \ln e = \frac{1}{x}\cdot 1$$

why?

$$= \frac{1}{x}.$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

The Derivative of e^x

Using a Trick and the Chain Rule:

Let $x = \ln e^x$ $\quad \log_e e^x = x$

(why?)

remember $\ln a = b$ means $\log_e a = b$

means $e^b = a$ $\quad$ $\ln e^x$ is a power that you raise e to to get e^x. That power is x.

$$\frac{d}{dx}(\ln(e^x)) = \frac{d}{dx}\, x \cdot \ln e$$

$$= \frac{d}{dx}\, x \cdot 1 = \frac{d}{dx}\, x = 1$$

$$\lim_{h \to 0} \frac{(x+h) - x}{h} = \frac{h}{h} = 1$$

But By the Chain Rule:

$$\frac{d}{dx}(\ln(e^x)) = \frac{1}{e^x} \cdot \frac{d}{dx} e^x$$

so $\quad 1 = \dfrac{1}{e^x}\,\dfrac{d}{dx}(e^x) \xrightarrow{e^x} \boxed{\dfrac{d}{dx}(e^x) = e^x}$ ■

The Power Rule

$$\boxed{\frac{d}{dx} x^n = n \cdot x^{n-1}}$$

Instead of $\displaystyle\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

By chain rule and a trick again! ☺

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(e^{\ln x^n}\right)$$

[$\ln x^n$ is a power you raise e to to get x^n. Here we are raising e to that power!]

Chain rule. Inner function.

$$\frac{d}{d*} e^* = e^*$$

$$\frac{d}{dx} \underbrace{e}_{\text{outer function}}^{\ln x^n} = \frac{d}{dx} e^{n \ln x} =$$

$$e^{\ln x^n} \cdot (n \ln x)' = x^n \cdot n \cdot \frac{1}{x}$$

$$= n x^{n-1}$$

Why?

remember $(\ln x)' = \frac{1}{x}$.

Now we can take

$$\frac{d\,(5x+4)^{100}}{dx} = 100(5x+4)^{99}\cdot 5$$

$(5x+4)' = 5$ ✓

$$= \boxed{500(5x+4)^{99}}$$

More examples

1. $(x^9)' = 9x^8$

2. $\frac{d}{dx}x = 1\cdot x^{1-1}$
$= 1\cdot x^0 = 1$

3. $\frac{d}{dx}(5) = \frac{d}{dx}5x^0 = 0\cdot 5\cdot x^{-1}$ $= 0$ (why makes sense?)

4. $\frac{d}{dx}(2^8) = 0$

why?

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5. $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x)^{1/2}$
$= \frac{1}{2}x^{\frac{1}{2}-\frac{2}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$
$= \boxed{\frac{1}{2\sqrt{x}}}$ ✓

6. $\frac{d}{dx}t^{100} = 0$ ← why?

The Product Rule

Recall $\frac{d}{dx}[f(x)] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

want $\frac{d}{dx}[f(x)\cdot g(x)] = \ldots$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

trick!

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\left(g(x+h)-g(x)\right) + g(x)\left(f(x+h)-f(x)\right)}{h}$$

$$= \lim_{h\to 0} \underset{\downarrow\, 0}{f(x+h)}\cdot\left[\frac{g(x+h)-g(x)}{h}\right] + g(x)\cdot\left[\frac{f(x+h)-f(x)}{h}\right]$$

as $h\to 0$

$\rightsquigarrow f(x)\cdot g'(x) + g(x)\cdot f'(x)$

$$\boxed{(fg)' = fg' + f'g}$$

The Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x)\cdot(g(x))^{-1}\right]' =$$

TRICK! → using the Product Rule

$$f'(x)\cdot\frac{1}{g(x)} + f(x)\underbrace{(-1)(g(x))^{-2}\cdot g'(x)}_{\text{power + chain rules.}}$$

$$= \frac{f'}{g} - \frac{fg'}{g^2}$$

$$= \frac{f'g}{g^2} - \frac{fg'}{g^2}$$

$$= \boxed{\frac{f'g - fg'}{g^2}}$$

ex. Find $\left[\frac{2x+3}{3x+1}\right]'$.